ELSEVIER

Discrete Mathematics 150 (1996) 441–447

DISCRETE MATHEMATICS

Note

On product of association schemes and Shannon capacity

Dezső Miklós[1]

Mathematical Institute of the Hungarian Academy of Sciences, P.O. Box 127, H-1364 Budapest, Hungary

Received 19 December 1993

Abstract

In this paper we will define the product of two association schemes and using the fact that the strong product of two graphs from two (possibly different) association schemes is in the product of the association schemes, we give a new proof of Schrijver's result on the Shannon capacity of graphs in association schemes. In particular, this will give a new proof of the fact that the Shannon capacity of the pentagon is $\sqrt{5}$.

1. Introduction

In 1956, Shannon [5] introduced the zero-error capacity of graphs in the following way: For any graph G, let $\alpha(G)$ denote the *independence number* of the graph, i.e. the maximum number of pairwise non-adjacent vertices.

The *weak, Cartesian and strong products* (denoted by $G \otimes H$, $G \times H$ and $G \cdot H$, resp.) of any two finite graphs G and H are defined to be the graphs whose vertex sets are always the Cartesian product of the vertex sets of G and H, $V(G) \times V(H)$ such that two vertices (g_1, h_1) and (g_2, h_2) are joined by an edge in

- $G \otimes H$ iff $g_1 \neq g_2$, $h_1 \neq h_2$, $(g_1, g_2) \in E(G)$ and $(h_1, h_2) \in E(H)$,
- $G \times H$ iff either $g_1 = g_2$, $h_1 \neq h_2$ and $(h_1, h_2) \in E(H)$ or $g_1 \neq g_2$, $h_1 = h_2$ and $(g_1, g_2) \in E(G)$,
- $G \cdot H$ iff $(g_1, h_1) \neq (g_2, h_2)$ and in both coordinates they are either equal or joined by an edge in G or H, resp.

Notice that the strong product of two graphs is just the union of the weak and Cartesian products of the same graphs. Also, the strong product of two cliques is always a clique, but this is not the case for the weak and Cartesian products.

[1] This research was partially supported by Hungarian National Foundation of Scientific Research Grant nos. 1812 and 4267.

0012-365X/96/$15.00 © 1996—Elsevier Science B.V. All rights reserved

SSDI 0012-365X(95)00213-8

With these definitions the *Shannon capacity* of a graph is defined by $\Theta(G)= \lim \sqrt[k]{\alpha(G^k)} = \limsup \sqrt[k]{\alpha(G^k)} = \sup \sqrt[k]{\alpha(G^k)}$ where the power of the graph G is taken according to the strong product. (For code-theoretical background of this notion see [2] or [5].) Throughout this paper the power of graphs will be taken according to the strong product.

Obviously, a trivial lower bound on the Shannon capacity of a graph G is its independence number $\alpha(G)$, but for the pentagon we have that $\alpha(C_5^2) = 5$ and so $\Theta(C_5) \geqslant \sqrt{5} > 2$. In general, it is much more difficult to find an upper bound for the Shannon capacity.

It is easy to see that any functional $\vartheta(G)$ defined on the class of finite graphs satisfying the properties

(a) $\alpha(G) \leqslant \vartheta(G)$ for all graphs G and

(b) $\vartheta(G^k) \leqslant \vartheta(G)^k$ for all graphs G

will be an upper bound on $\Theta(G)$ for we have $\alpha(G^k) \leqslant \vartheta(G^k) \leqslant \vartheta(G)^k$ and so $\Theta(G) \leqslant \sqrt[k]{\vartheta(G)^k} = \vartheta(G)$.

It is easy to see that for graphs whose vertex set can be covered by $\alpha(G)$ cliques the Shannon capacity coincides with $\alpha(G)$; in general, by the above remark, the fractional vertex packing $\alpha^*(G)$ of a graph G is always an upper bound for $\Theta(G)$. However, this still only gives us that $\sqrt{5} \leqslant \Theta(C_5) \leqslant \frac{5}{2}$.

In 1978 Lovász gave a powerful general upper bound on the Shannon capacity [2] and later in the same year Schrijver [4] and independently McEliece et al. [3] found alternative forms of the same bound for graphs which arise from association schemes. This bound, in particular, gave that $\Theta(C_5) = \sqrt{5}$. The aim of this paper is to show that by simply defining the product of association schemes one may obtain the same upper bound for graphs which are contained in association schemes.

2. Association schemes

A set of symmetric relations $R_0, R_1, \ldots, R_n$ on the set X or of symmetric 0–1 matrices $A_0, A_1, \ldots, A_n$ of size $|X| = v$ are called a *symmetric association scheme* and is denoted by $(X, \mathscr{R})$ or $(X, \mathscr{A})$, resp., if the relations or matrices satisfy the following properties:

(i) $R_0 = \{(x,x) \mid x \in X\}$ or $A_0 = I_{|X|}$,

(ii) for every $x, y \in X$ (not necessarily different) there is exactly one R_i such that xR_iy (and so yR_ix) or $\sum_{i=0}^{n} A_i = J_{|X|}$ where J denotes the all-one matrix,

(iii) for all $i, j, k \in \{0, 1, \ldots, n\}$ and $x, y \in X$ such that xR_ky we have that $|\{z \mid (x,y) \in R_i \text{ and } (y,z) \in R_j\}| = p_{ij}^k$ or $A_iA_j = \sum_{k=0}^{n} p_{ij}^k A_k$.

The set of the indices of the relations or matrices $\{0, 1, \ldots, n\}$ will always be denoted by N. A symmetric association scheme (from now on we always assume that an association scheme is symmetric) can always be viewed as a partition of the edge set of a complete graph into subgraphs which are defined by the above symmetric matrices $A_1, A_2, \ldots, A_n$. We will denote the corresponding graphs by $G_1, G_2, \ldots, G_n$ (we will ignore as a graph the one corresponding to A_0, i.e. the graph consisting of all the loops and nothing else) and in this case we denote the association scheme by $\mathscr{G}$. Thus, we may say that a graph G *arises from an association scheme* or *is in an association scheme* if there is an association scheme $\mathscr{G} = \{G_1, G_2, \dots, G_n\}$ and a set $M \subset N \setminus \{0\}$ such that $G = G_M = \bigcup_{i \in M} G_i$.

For our purpose the best examples of association schemes are just the graphs which together with their complement form an association scheme (with two classes besides the trivial one, i.e. with $n = 2$). They are called *strongly regular graphs.* Obviously, the pentagon and the complement of it, another pentagon, form an association scheme, so the pentagon is a strongly regular graph and thus arises from an association scheme. For other examples of association schemes see e.g. [1] or [4].

To investigate the independence number of graphs in association schemes an important tool is the *inner distribution* of a set $Y \subset X$ where X is the common vertex set of the graphs in the association scheme. Let e_Y be the characteristic vector of Y, then the inner distribution is defined to be the sequence $a_0, a_1, \dots, a_n$, where the numbers a_i are equal to $e_Y A_i e_Y^{\mathrm{t}}/|Y|$. One can easily see that $a_0 = 1$ and $\sum_{i=0}^{n} a_i = |Y|$ for all Y. (One may view a_i as the average degree of the graph $G_i|_Y$, the graph G_i restricted to the vertices belonging to Y.) Also, if $G_M = \bigcup_{i \in M} G_i$ is a graph from an association scheme and Y is an independent set of its vertices, then for the inner distribution of Y we have $a_i = 0$ for $i \in M$.

The vector space spanned by the matrices of an association scheme by property (iii) forms a commutative semi-simple algebra, and so besides the original basis $A_0, A_1, \dots, A_n$ it has another basis $J_0, J_1, \dots, J_n$ of orthonormal idempotent matrices, where we suppose that $J_0 = (1/v)J$, J being the all-one matrix. The transition numbers $P_k(i)$ and $Q_k(i)$, $0 \leqslant k \leqslant n$, $0 \leqslant i \leqslant n$ between the two bases of this algebra are defined by

$$A_k = \sum_{i=0}^{n} P_k(i) J_i, \quad 0 \leqslant k \leqslant n, \qquad J_k = \frac{1}{v} \sum_{i=0}^{n} Q_k(i) A_i, \quad 0 \leqslant k \leqslant n$$

where the normalization of $Q_k(i)$ by the factor $1/v$ is for technical purposes. The following basic theorem of Delsarte about the inner distribution gives a basic tool for investigating the independence number and the Shannon capacity of graphs arising from association schemes.

Theorem 1 (Delsarte [1]). *If in the association scheme $\mathscr{A}$ the vector $\mathbf{a} = (a_0, a_1, \dots, a_n)$ is the inner distribution of a set $Y \subset X$ then*

$$\sum_{i=0}^{n} a_i Q_j(i) \geqslant 0 \quad \textit{for all } 0 \leqslant j \leqslant n.$$

Proof. Let again e_Y denote the characteristic vector of Y, and so

$$\sum_{i=0}^{n} a_i Q_j(i) = \sum_{i=0}^{n} \frac{e_Y A_i e_Y^{\mathrm{t}}}{|Y|} Q_j(i) = \frac{1}{|Y|} e_Y \left(\sum_{i=0}^{n} Q_j(i) A_i \right) e_Y^{\mathrm{t}} = \frac{v}{|Y|} e_Y J_j e_Y^{\mathrm{t}}$$
$$= \frac{v}{|Y|} e_Y J_j J_j^{\mathrm{t}} e_Y^{\mathrm{t}} = \frac{v}{|Y|} (e_Y J_j)(e_Y J_j)^{\mathrm{t}} \geqslant 0. \qquad \square$$

With the help of this theorem, Delsarte established the following upper bound for the independence number of a graph in an association scheme.

Theorem 2 (Delsarte [1]). *If for some $M \subset N$, $G_M = \bigcup_{i \in M} G_i$ is a graph in the association scheme $\mathscr{A}$, then*

$$\alpha(G_M) \leqslant \max\left\{\sum_{i=0}^{n} a_i \mid a_i \geqslant 0,\ a_0 = 1,\ a_k = 0 \text{ for } k \in M \text{ and } \sum_{i=0}^{n} a_i Q_j(i) \geqslant 0 \text{ for all } 0 \leqslant j \leqslant n\right\}$$

$$= \min\left\{\sum_{j=0}^{n} b_j \mid b_j \geqslant 0,\ b_0 = 1,\ \sum_{j=0}^{n} b_j P_k(j) \leqslant 0 \text{ for all } k \in \overline{M} \cup \{0\}\right\}.$$

Proof. The first 'max' bound comes from the fact that if $Y \subset X$ is a maximum independent set of vertices of the graph G_M then its inner distribution $\boldsymbol{a} = (a_0, a_1, \ldots, a_n)$ will satisfy the linear program and the size of $Y = \sum_{i=0}^{n} a_i$, while the second one simply from the duality theorem of linear programming. □

3. Product of association schemes

If there are two association schemes $\mathscr{A}$ and $\mathscr{B}$ given by the matrices $A_0, A_1, \ldots, A_n$ and $B_0, B_1, \ldots, B_m$ on the ground sets X_1 and X_2, respectively, then the tensor product of the matrices of the form $A_i \otimes B_j$ — as one can see easily — form another association scheme on the ground set $X_1 \times X_2$, denoted by $\mathscr{A} \otimes \mathscr{B}$. Here, since the matrices A_i and B_j are just 0–1 matrices, one may take the tensor product $A_i \otimes B_j$ by simple plugging in a copy of B_j to the entries of A_i equal to one and an all-zero matrix of size $|X_2| \times |X_2|$ to the entries of A_i equal to zero, getting a matrix of size $|X_1| \cdot |X_2| \times |X_1| \cdot |X_2|$.

It is also trivial that if we have two graphs G_{M_1}, $M_1 \subseteq \{1, 2, \ldots n\}$ and G_{M_2}, $M_2 \subseteq \{1, 2, \ldots, m\}$ from the association schemes $\mathscr{A}$ and $\mathscr{B}$, resp., then the weak product $G_{M_1} \otimes G_{M_2}$ will be exactly equal to the graph $G_{M_1 \times M_2}$ from the product of the association schemes, the Cartesian product will be $G_{\{0\} \times M_2} \cup G_{M_1 \times \{0\}}$ and the strong product will be the union of the above, that is

$$G \cdot H = G_{M_1 \times M_2} \cup G_{\{0\} \times M_2} \cup G_{M_1 \times \{0\}},$$

so all the possible products are in the product association scheme.

Tedious, but trivial calculations show that if we take the product of two association schemes $\mathscr{A}$ and $\mathscr{B}$, then the parameters of the product association scheme (p_{ij}^k and the transition numbers $P_k(i)$ and $Q_k(i)$) will be formed by the product of the corresponding parameters of the original association schemes. In particular, the transition numbers involved in Theorems 1 and 2 will be in the form

$$A_k \otimes B_l = \left(\sum_{i=0}^{n} P_k(i)J_i\right) \otimes \left(\sum_{j=0}^{m} P_l'(j)K_j\right)$$
$$= \sum_{\substack{i=0\\j=0}}^{n,m} P_k(i)P_l'(j)(J_i \otimes K_j), \quad 0 \leqslant k \leqslant n, \ 0 \leqslant l \leqslant m$$

$$J_k \otimes K_l = \left(\frac{1}{v}\sum_{i=0}^{n} Q_k(i)A_i\right) \otimes \left(\frac{1}{w}\sum_{j=0}^{m} Q_l'(j)B_j\right)$$
$$= \frac{1}{vw}\sum_{\substack{i=0\\j=0}}^{n,m} Q_k(i)Q_l'(j)(A_i \otimes B_j), \quad 0 \leqslant k \leqslant n, \ 0 \leqslant l \leqslant m,$$

where

— as mentioned earlier, the matrices $A_k \otimes B_l$ define the product association scheme,

— the matrices J_i and K_j form the bases of idempotent orthonormal matrices in $\mathscr{A}$ and $\mathscr{B}$, resp., and so their tensor products, $J_i \otimes K_j$ form the bases of idempotent orthonormal matrices in $\mathscr{A} \otimes \mathscr{B}$, and

— $|X_1| = v, \ |X_2| = w$.

4. Shannon capacity of graphs in association schemes

We are now ready to define a functional satisfying properties (a) and (b) *on the subclass of graphs which arise from an association scheme.* Since, as mentioned earlier, this subclass is closed under the strong product operation, the properties are meaningful.

Unfortunately, the straightforward bound

$$\alpha(G_M) \leqslant \max\left\{\sum_{i=0}^{n} a_i \mid \underline{a_i \geqslant 0}, \ a_0 = 1, \ a_k = 0 \ \forall k \in M, \ \sum_{i=0}^{n} a_i Q_j(i) \geqslant 0 \ \forall j\right\}$$
$$= \min\left\{\sum_{j=0}^{n} b_j \mid b_j \geqslant 0, \ b_0 = 1, \ \underline{\sum_{j=0}^{n} b_j P_k(j) \leqslant 0 \text{ for all } k \in \overline{M} \cup \{0\}}\right\}$$

will not satisfy property (b), for the underlined inequality in the 'min' bound will not be preserved by the product. However, if we change this bound to

$$\alpha(G_M) \leqslant \vartheta(G_M)$$
$$\stackrel{\text{def}}{=} \max\left\{\sum_{i=0}^{n} a_i \mid a_0 = 1, \ a_k = 0 \ \forall k \in M, \sum_{i=0}^{n} a_i Q_j(i) \geqslant 0 \ \forall j\right\}$$
$$= \min\left\{\sum_{j=0}^{n} b_j \mid b_j \geqslant 0, \ b_0 = 1, \ \sum_{j=0}^{n} b_j P_k(j) = 0 \text{ for all } k \in \overline{M} \cup \{0\}\right\}$$

(where the earlier underlined positivity constraint has been left out from the first 'max' program, and so in the dual 'min' program the earlier underlined inequality $\leqslant$ has been changed to equality) this new linear programming bound will satisfy the properties (a) and (b) and so will be an upper bound on the Shannon capacity.

Theorem 3 (Mceliece et al. [3] and Schrijver [4]). *For a graph G_M which is in an association scheme, we have*

$$\Theta(G_M)\leqslant\vartheta(G_M)$$
$$\stackrel{\text{def}}{=} \max\left\{\sum_{i=0}^{n}a_i \mid a_0=1,\ a_k=0\ \forall k\in M,\ \sum_{i=0}^{n}a_iQ_j(i)\geqslant 0\ \forall j\right\}$$
$$=\min\left\{\sum_{j=0}^{n}b_j \mid b_j\geqslant 0,\ b_0=1,\ \sum_{j=0}^{n}b_jP_k(j)=0 \textit{ for all } k\in\overline{M}\cup\{0\}\right\}.$$

Proof. We prove that the functional $\vartheta(G)$ defined on the class of graphs which are in some association scheme satisfies properties (a) and (b).

Property (a), that is $\Theta(G)\leqslant\vartheta(G)$ is trivial from the above remarks. For property (b), let G_{M_1} and G_{M_2} be two graphs from the association schemes $\mathscr{A}$ and $\mathscr{B}$, respectively, and

$$\vartheta(G_{M_1})=\left\{\sum_{i=0}^{n}b_i \mid b_i\geqslant 0,\ b_0=1,\ \sum_{i=0}^{n}b_iP_k(i)=0 \text{ for all } k\in\overline{M_1}\cup\{0\}\right\},$$
$$\vartheta(G_{M_2})=\left\{\sum_{j=0}^{m}b'_j \mid b'_j\geqslant 0,\ b'_0=1,\ \sum_{j=0}^{m}b'_jP'_l(j)=0 \text{ for all } l\in\overline{M_2}\cup\{0\}\right\},$$

where the values of b_i's and b'_j's are choosen such a way that there sum is minimal for the given set of constraints. Now for the minimal program for the graph $G_N := G_{M_1}\cdot G_{M_2} = G_{M_1\times M_2}\cup G_{\{0\}\times M_2}\cup G_{M_2\times\{0\}}$ from the product association scheme $\mathscr{A}\otimes\mathscr{B}$:

$$\vartheta(G_N)=\min\left\{\sum_{\substack{i=0\\j=0}}^{n,m}c_{ij}\middle| c_{ij}\geqslant 0,\ c_{00}=1,\ \sum_{\substack{i=0\\j=0}}^{n,m}c_{ij}P_k(i)P'_l(j)=0\right.$$
$$\left.\text{for all } (k,l)\notin\big((M_1\times M_2)\cup(\{0\}\times M_2)\cup(M_2\times\{0\})\big)\right\}$$

define $c_{ij} := b_i\cdot b'_j$. The constraints are straightforwardly satisfied:
— since both b_i and $b'_j\geqslant 0$ we have that $c_{ij}=b_i\cdot b'_j\geqslant 0$;
— since both b_0 and $b'_0=0$ we have that $c_{00}=b_0\cdot b'_0=0$;
— we have that $\sum_{i=0}^{n}b_iP_k(i)=0$ for all $k\notin M_1$ and $\sum_{j=0}^{m}b'_jP'_l(j)=0$ for all $l\notin M_2$ and so $\sum_{i,j=0}^{n,m}c_{ij}P_k(i)P'_l(j) = \sum_{i,j=0}^{n,m}b_ib'_jP_k(i)P'_l(j) = \sum_{i=0}^{n}b_iP_k(i)\cdot\sum_{j=0}^{m}b'_j P'_l(j)=0$ whenever either $k\notin M_1$ or $l\notin M_2$.

Thus, the minimal program has value $\vartheta(G_N)$ smaller than or equal to

$$\sum_{\substack{i=0\\j=0}}^{n,m} b_i b'_j = \sum_{i=0}^{n} b_i \cdot \sum_{j=0}^{m} b'_j = \vartheta(G_{M_1}) \cdot \vartheta(G_{M_2})$$

and so it is proven that $\vartheta(G_N) = \vartheta(G_{M_1} \cdot G_{M_2}) \leqslant \vartheta(G_{M_1}) \cdot \vartheta(G_{M_2})$. $\square$

Remark. A similar argument to the above one for the 'max' program shows that $\vartheta(G_N) \geqslant \vartheta(G_{M_1}) \cdot \vartheta(G_{M_2})$ as well, and so the bound $\vartheta(G)$ is multiplicative.

References

[1] P. Delsarte, An algebraic approach to the association schemes in coding theory, Philips Res. Reps. Suppl. 10 (1973).

[2] L. Lovász, On the Shannon capacity of a graph, IEEE Trans. Inform. Theory 25 (1979) 1–7.

[3] R.J. McEliece, E.R. Rodemich and H.C. Rumsey, The Lovász bound and some generalizations, J. Combin. Inform. System Sci. 3 (1978) 134–152.

[4] A. Schrijver, A comparision of the Delsarte and Lovász bounds, IEEE Trans. Inform. Theory 25 (1979) 425–429.

[5] C.E. Shannon, The zero-error capacity of a noisy channel, IRE Trans. Inform. Theory 3 (1956) 3–15.